Fourier Analysis 03-09.

Review

- Construction a cts function on the circle with diverging Fourier series
- A proof of the isoperimetric inequality using Fourier series.

§4.3 Weyl's equidistribution theorem

Def. A sequence of numbers $(x_n)_{n=1}^{\infty}$ in $[0,1)$ is said to be equidistributed in $[0,1)$ if the following holds: $\forall\ (a,b) \subset [0,1)$,

$$① \quad \lim_{N\to\infty} \frac{1}{N} \# \{ 1 \le n \le N : \quad x_n \in (a,b) \} = b-a.$$

Example: Consider $(x_n)_{n=1}^{\infty}$ given by

$$0, \tfrac{1}{2}, 0, \tfrac{1}{2}, 0, \tfrac{1}{2}, \cdots$$

The proportion of (x_n) in $(0, \frac{1}{3})$ is $0 \neq \frac{1}{3}$

Thm 1 (Weyl) Let γ be an irrational number.

Then the sequence $(\{n\gamma\})_{n=1}^{\infty}$ is equidistributed in $[0,1)$. Here $\{x\}$ denotes the fractional part of x.

Remark: By a thm of Kronecker, $(\{n\gamma\})_{n=1}^{\infty}$ is dense in $[0,1)$.

For $(a,b) \subset [0,1)$, let $\chi_{(a,b)} : [0,1) \to \mathbb{R}$ be defined as

$$\chi_{(a,b)}(x) = \begin{cases} 1 & \text{if } x \in (a,b) \\ 0 & \text{if } x \in [0,1) \setminus (a,b). \end{cases}$$

It is called the characteristic function of (a,b).

We extend $\chi_{(a,b)}$ to be 1-periodic function on $\mathbb{R}$.

applying to $\{n\gamma\}$

Now ① is equivalent to

$$② \quad \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \chi_{(a,b)}(n\gamma) = \int_0^1 \chi_{(a,b)}(x)\, dx$$

In order to prove ②, we first prove

Lemma 2. Let f be a 1-period cts function on $\mathbb{R}$. Then

$$③ \quad \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(n\gamma) = \int_0^1 f(x)\,dx.$$

Pf. Step 1: We prove ③ holds if f is of the form

$$f(x) = e^{2\pi i k x}, \quad k \in \mathbb{Z}.$$

When $k=0$, $f \equiv 1$, ③ holds in this case

For $k \neq 0$,

$$\frac{1}{N} \sum_{n=1}^{N} e^{2\pi i k n \gamma} = \frac{1}{N} \sum_{n=1}^{N} \left(e^{2\pi i k \gamma}\right)^n$$

$$= \frac{e^{2\pi i k \gamma}}{N} \cdot \frac{1-e^{2\pi i k N \gamma}}{1-e^{2\pi i k \gamma}}$$

So $$\left| \frac{1}{N} \sum_{n=1}^{N} e^{2\pi i k n \gamma} \right| \leq \frac{2}{N} \cdot \frac{1}{|1-e^{2\pi i k \gamma}|}$$

$\to 0$ as $N \to +\infty$.

<u>Step 2</u>: Notice that if f, g satisfy ③, then so is $\alpha f + \beta g$ with $\alpha, \beta \in \mathbb{C}$.

Hence ③ holds when f is a trigonometric polynomial $\sum_{|n| \le N} c_n e^{2\pi i n x}$ on $[0,1)$

step 3. Let f be an arbitrary 1-periodic cts function

Then by Weierstrass Approximation Thm, $\exists$ a trigonometric polynomial g s.t

$$|f(x) - g(x)| \le \varepsilon \quad \forall x \in \mathbb{R}.$$

Then

$$\left| \frac{1}{N} \sum_{n=1}^{N} f(n\gamma) - \int_0^1 f(x)\,dx \right|$$

$$\le \frac{1}{N} \sum_{n=1}^{N} |f(n\gamma) - g(n\gamma)| + \int_0^1 |f(x) - g(x)|\,dx$$

$$+ \left| \frac{1}{N} \sum_{n=1}^{N} g(n\gamma) - \int_0^1 g(x)\,dx \right|$$

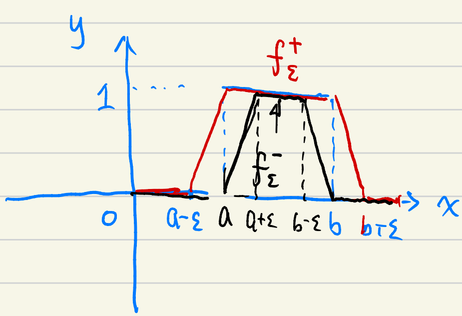
$$\leq \quad 2\varepsilon + \left| \frac{1}{N} \sum_{n=1}^{N} g(n\gamma) - \int_0^1 g(x)\,dx \right|$$

$$\leq \quad 3\varepsilon \quad \text{when } N \text{ is large enough (by step 2)}$$

$\blacksquare$

Pf of Weyl's Thm

Fix $(a,b) \subset [0,1)$.



Let $\varepsilon > 0$. We construct two cts functions, say f_ε^+, f_ε^- as above such that

$$f_\varepsilon^- \leq \chi_{(a,b)} \leq f_\varepsilon^+$$

and $$\int_0^1 f_\varepsilon^+ - \chi_{(a,b)}\, dx < 2\varepsilon$$

$$\int_0^1 \chi_{(a,b)} - f_\varepsilon^-\, dx < 2\varepsilon$$

Now

$$\overline{\lim_{N\to\infty}} \frac{1}{N}\sum_{n=1}^{N} \chi_{(a,b)}(n\gamma)$$

$$\leq \overline{\lim_{N\to\infty}} \frac{1}{N}\sum_{n=1}^{N} f_\varepsilon^+(n\gamma)$$

$$= \int_0^1 f_\varepsilon^+\, dx \leq \int_0^1 \chi_{(a,b)}(x)\, dx + 2\varepsilon$$

Similarly

$$\underline{\lim_{N\to\infty}} \frac{1}{N}\sum_{n=1}^{N} \chi_{(a,b)}(n\gamma)$$

$$\geq \underline{\lim_{N\to\infty}} \frac{1}{N}\sum_{n=1}^{N} f_\varepsilon^-(n\gamma)$$

$$= \int_0^1 f_\varepsilon^-\, dx \geq \int_0^1 \chi_{(a,b)}\, dx - 2\varepsilon$$

Since ε is arbitrarily taken, we see that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \chi_{(a,b)}(n\gamma) = \int_0^1 \chi_{(a,b)}(x)\,dx$$

$\square$.